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Liouville type theorem for a class quasilinear *p*-Laplace type equation on the sphere

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Abstract

We get the rigidity results for a class quasilinear p-Laplace type equation on the sphere. Rigidity means that the elliptic equation has no other solution than some constants at least when a parameter is in a certain range. This p-Laplace type equation arises from the study of asymptotic behavior near the origin for the semilinear p-Laplace equation on the punctured ball $B_1(o) \setminus \{o\} \subset \mathbb{R}^{n+1}$. Our result gives a positive answer to L. Véron's question in a paper (Véron in A geometric and analytic approach to some problems associated with Emden equations. Partial differential equations, Part 1, 2 (Warsaw, 1990), Banach Center Publ., 27, Part 1, 2. Polish Acad. Sci. Inst. Math., Warsaw, 1992) and his book (Véron in Local and global aspects of quasilinear degenerate elliptic equations: quasilinear elliptic singular problems. World Scientific, Singapore, 2017) at page 440.

1 Introduction

In 1981, Gidas and Spruck [9] studied the Liouville type properties of nonnegative solutions of the following semilinear elliptic equation

$$\Delta u + u^q = 0 \quad \text{in} \quad \mathbb{R}^{n+1}, \tag{1.1}$$

in the range of $1 < q < 2^* - 1$ where $2^* = \frac{2(n+1)}{n-1}$, they obtained that the unique solution must be the trivial one via the method of vector fields motivated by Obata [12].

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In order to study the asymptotic behavior near the origin for the above Eq. (1.1) on the punctured ball $B_1(o)\backslash\{o\} \subset \mathbb{R}^{n+1}$, Gidas and Spruck [9, Theorems B.1 and B.2] also investigated the following equation on the sphere $\mathbb{S}^n$,

$$\Delta u + u^q - \lambda u = 0 \quad \text{on} \qquad \mathbb{S}^n. \tag{1.2}$$

For the Eq. (1.2), under certain conditions on q and λ, they proved the constant $\lambda^{\frac{1}{q-1}}$ is the unique solution.

For the above semilinear equations on compact Riemannian manifolds, Bidaut-Véron and Véron [3, Theorem 6.1] introduced the Bochner–Lichnerowicz–Weitzenböck formula in such a way that they could extend and simplify Gidas and Spruck's results.

Theorem 1.1 (Bidaut–Véron–Véron [3]) *Assume* $(\mathbb{M}^n, g)$ *is a compact Riemannian manifold without boundary of dimension* $n \geq 2$, Δ *is the Laplace-Beltrami operator on* $\mathbb{M}^n$, $q > 1, \lambda > 0$ *and* u *is a positive solution of*

$$\Delta u + u^q - \lambda u = 0 \quad on \qquad \mathbb{M}^n. \tag{1.3}$$

Assume also that the spectrum $\sigma(R(x))$ *of the Ricci tensor* R *of the metric* g *satisfies*

$$\inf_{x\in\mathbb{M}^n} \min \sigma(R(x)) \geq \frac{n-1}{n}(q-1)\lambda, \quad q \leq \frac{n+2}{n-2}. \tag{1.4}$$

Moreover, assume that one of the two inequalities is strict if $(\mathbb{M}^n, g)$ *is conformally diffeomorphism to* $(\mathbb{S}^n, g_0)$. *Then* u *is constant with the value* $\lambda^{\frac{1}{q-1}}$.

Such a rigidity result has been extended in [10] and [11, Theorem 2.1] by Licois and Véron, and in [2, Inequality (1.11)] by Barky and Ledoux. Each of these contributions relies either on the Bochner–Lichnerowicz–Weitzenböck formula or on the carré du champ method. In Dolbeault et al. [7] (see also [8]), they gave a new approach relies on a nonlinear flow of porous medium/fast diffusion type which gives a clear-cut interpretation of technical choices of exponents done in earlier works on rigidity.

Now we turn to the following semilinear p-Laplace equation

$$\Delta_p u + u^q = 0 \quad \text{in} \qquad \mathbb{R}^{n+1}. \tag{1.5}$$

In 2002, Serrin and Zou [14] proved that nonnegative solutions of Eq. (1.5) must be zero by introducing a new vector field, for $1 < p < n+1$ and $p-1 < q < p^*-1$ where $p^* = \frac{(n+1)p}{n+1-p}$. Recently, using the differential identity of [14], Ciraolo et al. [5] classified the positive energy finite solutions to (1.5) when $q = \frac{(n+1)p}{n+1-p} - 1$ in convex cones with the help of some a prior estimates. One can find the recent results for the critical p-Laplace equation in $\mathbb{R}^n$ by Ciraolo and Corso [4] and Ou [13]. In Ciraolo et al. [5], an important Lemma in [1, 6] for the research of p-Laplacian equations has been used.

Now it is natural to study the rigidity result for the p-Laplace type equation on compact manifold.

In order to study the asymptotic behavior near the origin for Eq. (1.5) on the punctured ball $B_1(o)\backslash\{o\} \subset \mathbb{R}^{n+1}$, Véron [16] made the following observation. With the spherical coordinate (r, σ), separable solutions of (1.5) under the form $u(x) = u(r, \sigma) = r^{-\alpha}\omega(\sigma)$ exist, then ω satisfies

$$div\left(\left(\alpha_{p,q}^2\omega^2 + |\nabla\omega|^2\right)^{\frac{p-2}{2}}\nabla\omega\right) + |\omega|^{q-1}\omega$$
$$-\lambda_{p,q}\left(\alpha_{p,q}^2\omega^2 + |\nabla\omega|^2\right)^{\frac{p-2}{2}}\omega = 0 \quad \text{on} \qquad \mathbb{S}^n, \tag{1.6}$$

where

$$\lambda_{p,q} = \alpha_{p,q}(n+1-\alpha_{p,q}q), \qquad \alpha = \alpha_{p,q} = \frac{p}{q+1-p},$$

div and ∇ are operators under the canonical metric on $\mathbb{S}^n$.

If $\lambda_{p,q} \le 0$, i.e. $p-1 < q \le \frac{(n+1)(p-1)}{n+1-p}$; integrating equations (1.6) shows that there exists no nontrivial solution to (1.6).

For $q = \frac{np-n+p}{n-p}$, which is the Sobolev critical exponent, it was observed by Véron (see page 368 in [16]) that (1.6) admits nonconstant solutions.

For $\lambda_{p,q} > 0$, and $\frac{(n+1)(p-1)}{n+1-p} < q < \frac{np-n+p}{n-p}$, then for the Eq. (1.6) the positive constant solution is

$$\Lambda_{n,p,q} = (\alpha_{p,q}^{p-1}(n+1-\alpha_{p,q}q))^{\frac{1}{q+1-p}}.$$

In a paper [15] and his book [16] at page 440, Véron asked if all **positive** solutions of (1.6) are constants $\Lambda_{n,p,q}$. In this paper, we give a positive answer to it.

Theorem 1.2 *For $1 < p < n$ and $\frac{(n+1)(p-1)}{n+1-p} < q < \frac{np-n+p}{n-p}$ with $\alpha_{p,q} = \frac{p}{q+1-p}$ and $\lambda_{p,q} = \alpha_{p,q}(n+1-\alpha_{p,q}q)$, any positive solution to (1.6) is constant $\Lambda_{n,p,q}$.*

In the research of p-Laplace equation [4, 13, 14], one always use integral by parts method through introducing some parameters to get these results. In the proof of our Theorem 1.2, we introduce three parameters, especially in (2.3) we add a new term in the usual trace free term E^i_j, which has not appeared in the earlier works on rigidity. At last we use the Lemma from [1] or [6] to complete our proof.

The paper is organized as follows. In Sect. 2, we introduce some notations and prove an integral equality. Then we use the integral equality through choosing these parameters to prove Theorem 1.2 in Sect. 3.

2 An integral equality

In this section,we drive a useful equality.

Let $\omega = v^{-\beta}$, $\beta \neq 0$. We denote $k = (\beta+1)(p-1)-\beta q$, $Q = (\alpha^2 v^2+\beta^2|\nabla v|^2)^{\frac{1}{2}}$, $X^i = Q^{p-2}v_i$, $X^i_j = (Q^{p-2}v_i)_j$, $E^i_j = X^i_j - \frac{div_g(X^l)}{n}g_{ij}$, $L^i_j = Q^{p-2}(\frac{v_i v_j}{v} - \frac{|\nabla v|^2}{nv}g_{ij})$. Then

$$E^i_j E^j_i = X^i_j X^j_i - \frac{X^i_i X^j_j}{n}. \tag{2.1}$$

and

$$|L^i_j|^2 = \frac{n-1}{n} Q^{2p-4} v^{-2} |\nabla v|^4 \tag{2.2}$$

We modify E^i_j to deal with the subcritical case. We set

$$F^i_j = E^i_j + \varepsilon div_g(X^l) g_{ij}, \tag{2.3}$$

for some $\varepsilon \neq 0$, which is $F^i_j = X^i_j + (\varepsilon - \frac{1}{n}) div_g(X^l) g_{ij}$. Using the fact that L^i_j is trace free, we have

$$F^i_j F^j_i = X^i_j X^j_i + \left(n\varepsilon^2 - \frac{1}{n}\right)(X^l_l)^2, \tag{2.4}$$

$$F^i_j L^i_j = E^i_j L^i_j = Q^{p-2}\left[(Q^{p-2} v_i)_j \frac{v_i v_j}{v} - (Q^{p-2} v_l)_l \frac{|\nabla v|^2}{nv}\right]. \tag{2.5}$$

Then our Eq. (1.6) becomes

$$X^i_i - (\beta+1)(p-1) v^{-1} |\nabla v|^2 Q^{p-2} - \beta^{-1} v^k + \beta^{-1} \lambda Q^{p-2} v = 0. \tag{2.6}$$

Multiplying (2.6) with $v^a X^j_j$ and integrating on $\mathbb{S}^n$, we get

$$\int v^a X^i_i X^j_j - (\beta+1)(p-1) \int v^{a-1} |\nabla v|^2 Q^{p-2} X^j_j - \beta^{-1} \int v^{k+a} X^j_j$$
$$+ \beta^{-1} \lambda \int v^{a+1} Q^{p-2} X^j_j = 0, \tag{2.7}$$

where $a \neq 0$ will be determined later.

After integrating by parts, we deal with the third term in (2.7) directly

$$-\beta^{-1} \int v^{k+a} X^j_j = \beta^{-1}(k+a) \int v^{k+a-1} |\nabla v|^2 Q^{p-2}. \tag{2.8}$$

Note that

$$(Q^{p-2})_j = \left[(\alpha^2 v^2 + \beta^2 |\nabla v|^2)^{\frac{p-2}{2}}\right]_j = (p-2) Q^{p-4} (\alpha^2 v v_j + \beta^2 v_l v_{lj}). \tag{2.9}$$

To be convenient, we set $f = vv_jv_iv_{ji} - |\nabla v|^4$, then the last term in (2.7) becomes

$$\begin{aligned}
&\beta^{-1}\lambda\int v^{a+1}Q^{p-2}(Q^{p-2}v_j)_j\\
&= -\beta^{-1}\lambda(a+1)\int v^a|\nabla v|^2Q^{2p-4} - \beta^{-1}\lambda\int v^{a+1}(Q^{p-2})_jQ^{p-2}v_j\\
&= -\beta^{-1}\lambda(a+p-1)\int v^a|\nabla v|^2Q^{2p-4} - (p-2)\beta\lambda\int v^aQ^{2p-6}f.
\end{aligned} \tag{2.10}$$

For the first term in (2.7), we observe that $(Q^{p-2}v_j)_{ji} = (Q^{p-2}v_j)_{ij} - R_{ji}v_jQ^{p-2}$ where R_{ij} is the Ricci curvature. So we have

$$\begin{aligned}
\int v^a(Q^{p-2}v_i)_iX_j^j =& -a\int v^{a-1}Q^{p-2}|\nabla v|^2X_j^j - \int v^aQ^{p-2}v_iX_{ji}^j\\
=& -a\int v^{a-1}Q^{p-2}|\nabla v|^2X_j^j - \int v^a(Q^{p-2}v_j)_ij\,Q^{p-2}v_i\\
&+\int v^aQ^{2p-4}R_{ji}v_jv_i\\
=& -a\int v^{a-1}Q^{p-2}|\nabla v|^2X_j^j + \int v^a(Q^{p-2}v_i)_j(Q^{p-2}v_j)_i\\
&+a\int v^{a-1}(Q^{p-2}v_j)_iQ^{p-2}v_iv_j + \int v^aQ^{2p-4}R_{ji}v_jv_i.
\end{aligned}$$

Invoking (2.4), the first term in (2.7) becomes

$$\begin{aligned}
\int v^aX_i^iX_j^j =& -\frac{na}{n-1+n^2\varepsilon^2}\int v^{a-1}Q^{p-2}|\nabla v|^2X_i^i + \frac{n}{n-1+n^2\varepsilon^2}\int v^aR_{ji}v_jv_i\\
&+\frac{na}{n-1+n^2\varepsilon^2}\int v^{a-1}(Q^{p-2}v_j)_iQ^{p-2}v_iv_j\\
&+\frac{n}{n-1+n^2\varepsilon^2}\int v^aF_j^iF_i^j.
\end{aligned} \tag{2.11}$$

The term $v^{a+k-1}|\nabla v|^2Q^{p-2}$ in (2.8) appears for the reason that the equation is not homogeneous. It is desirable to eliminate it with some equalities. To deal with it, we multiply the Eq. (2.6) with $|\nabla v|^2v^{a-1}Q^{p-2}$. Then for the third term in (2.7), we get

$$\begin{aligned}
-\beta^{-1}\int v^{k+a}X_j^j =& \beta^{-1}(k+a)\int v^{a+k-1}|\nabla v|^2Q^{p-2}\\
=& (k+a)\int v^{a-1}|\nabla v|^2Q^{p-2}X_i^i - (k+a)(\beta+1)(p-1)\\
&\int v^{a-2}|\nabla v|^4Q^{2p-4}
\end{aligned}$$

$$+ (k+a)\beta^{-1}\lambda \int v^a |\nabla v|^2 Q^{2p-4}. \tag{2.12}$$

Recalling that $k = (\beta+1)(p-1) - \beta q$ and $R_{ij} = (n-1)g_{ij}$, combining (2.10), (2.11) and (2.12) with (2.7), we arrive at the following integral identity.

Proposition 2.1 *If v is a positive solution of the Eq. (2.6), then we have*

$$\begin{aligned}
&\left(-\beta q + \frac{-a + an^2\varepsilon^2}{n-1+n^2\varepsilon^2}\right) \int v^{a-1} Q^{p-2} |\nabla v|^2 X_i^i \\
&+ \frac{na}{n-1+n^2\varepsilon^2} \int v^{a-1} Q^{p-2} v_i v_j (Q^{p-2} v_i)_j \\
&+ \left[\frac{n(n-1)}{n-1+n^2\varepsilon^2} + \lambda(p-1-q)\right] \int v^a |\nabla v|^2 Q^{2p-4} + \frac{n}{n-1+n^2\varepsilon^2} \int v^a F_j^i F_i^j \\
&-(k+a)(\beta+1)(p-1) \int v^{a-2} |\nabla v|^4 Q^{2p-4} - \beta\lambda(p-2) \int v^a Q^{2p-6} f = 0.
\end{aligned} \tag{2.13}$$

To address the last term for $p \neq 2$, we need the following lemma.

Lemma 2.2 *We have*

$$\begin{aligned}
\int v^{a-1} Q^{p-2} |\nabla v|^2 (Q^{p-2} v_i)_i &= -(a-1) \int v^{a-2} |\nabla v|^4 Q^{2p-4} \\
&- \frac{p}{p-1} \int v^{a-1} (Q^{p-2} v_j)_i Q^{p-2} v_i v_j - \frac{p-2}{p-1} \alpha^2 \int v^a Q^{2p-6} f.
\end{aligned} \tag{2.14}$$

Proof Combining

$$\begin{aligned}
L.H.S &= \int v^{a-1} Q^{p-2} v_j v_j (Q^{p-2} v_i)_i \\
&= -(a-1) \int Q^{2p-4} v^{a-2} |\nabla v|^4 - \int v^{a-1} (Q^{p-2} v_j)_i Q^{p-2} v_i v_j \\
&\quad - \int v^{a-1} Q^{2p-4} v_i v_j v_{ij}
\end{aligned}$$

and

$$\begin{aligned}
&(Q^{p-2} v_j)_i Q^{p-2} v_i v_j \\
&= Q^{2p-4} v_{ji} v_i v_j + (p-2) Q^{2p-6} (\alpha^2 v |\nabla v|^4 + \beta^2 |\nabla v|^2 v_i v_k v_{ki}) \\
&= Q^{2p-4} v_{ji} v_i v_j - (p-2)\alpha^2 v Q^{2p-6} f + (p-2) Q^{2p-6} v_i v_k v_{ik} (\alpha^2 v^2 + \beta^2 |\nabla v|^2)
\end{aligned}$$

$$= (p-1)Q^{2p-4}v_{ji}v_iv_j - (p-2)\alpha^2 v Q^{2p-6} f$$

we get (2.14). □

Therefore the last term in (2.13) becomes

$$\begin{aligned} -\beta\lambda(p-2)\int v^a Q^{2p-6} f = &\frac{\beta\lambda(p-1)}{\alpha^2}\int v^{a-1}Q^{p-2}|\nabla v|^2 X_i^i \\ &+\frac{\beta\lambda(p-1)(a-1)}{\alpha^2}\int v^{a-2}|\nabla v|^4 Q^{2p-4} \\ &+\frac{\beta\lambda p}{\alpha^2}\int v^{a-1}(Q^{p-2}v_j)_i Q^{p-2}v_iv_j. \end{aligned} \tag{2.15}$$

We come to the following important integral identity.

Proposition 2.3 *If v is a positive solution of the Eq. (2.6), then for any constants ε, β, a, we have*

$$\begin{aligned} 0 = &\left[-\beta q + \frac{-a+an^2\varepsilon^2}{n-1+\varepsilon^2 n^2} + \frac{\beta\lambda(p-1)}{\alpha^2}\right]\int v^{a-1}Q^{p-2}|\nabla v|^2 X_i^i \\ &+\left[\frac{n(n-1)}{n-1+n^2\varepsilon^2} + \lambda(p-1-q)\right]\int v^a|\nabla v|^2 Q^{2p-4} \\ &+\left(\frac{\beta\lambda p}{\alpha^2} + \frac{na}{n-1+n^2\varepsilon^2}\right)\int v^{a-1}Q^{p-2}v_iv_j(Q^{p-2}v_i)_j \\ &+\left[\frac{\beta\lambda(p-1)(a-1)}{\alpha^2} - (k+a)(\beta+1)(p-1)\right]\int v^{a-2}|\nabla v|^4 Q^{2p-4} \\ &+\frac{n}{n-1+n^2\varepsilon^2}\int v^a F_j^i F_i^j. \end{aligned} \tag{2.16}$$

3 Proof of the Theorem 1.2

In this section, through the choice of the constants ε, β, a, we analyze the coefficients in (2.16). We prove $|\nabla v| = 0$, which implies $|\nabla\omega| = 0$, then we complete the proof of our Theorem 1.2.

Using (2.5) in the third term in (2.16), we can rewrite (2.16) as

$$\begin{aligned} 0 = &\left[-\beta q + \frac{-a+an^2\varepsilon^2}{n-1+\varepsilon^2 n^2} + \frac{\beta\lambda(p-1)}{\alpha^2}\right]\int v^{a-1}Q^{p-2}|\nabla v|^2 X_i^i \\ &+\left[\frac{n(n-1)}{n-1+n^2\varepsilon^2} + \lambda(p-1-q)\right]\int v^a|\nabla v|^2 Q^{2p-4} \\ &+\left(\frac{\beta\lambda p}{\alpha^2} + \frac{na}{n-1+n^2\varepsilon^2}\right)\int v^a F_j^i L_j^i \\ &+\left(\frac{\beta\lambda p}{n\alpha^2} + \frac{a}{n-1+n^2\varepsilon^2}\right)\int v^{a-1}Q^{p-2}|\nabla v|^2 X_i^i \end{aligned}$$

$$+\left[\frac{\beta\lambda(p-1)(a-1)}{\alpha^2}-(k+a)(\beta+1)(p-1)\right]\int v^{a-2}|\nabla v|^4 Q^{2p-4}$$
$$+\frac{n}{n-1+n^2\varepsilon^2}\int v^a F^i_j F^j_i. \tag{3.1}$$

To be convenient, we set

$$M=\frac{\frac{\beta\lambda p}{\alpha^2}+\frac{na}{n-1+n^2\varepsilon^2}}{\frac{2n}{n-1+n^2\varepsilon^2}}.$$

By (2.2), we get the following crucial integral identity.

$$0=\left[-\beta q+\frac{-a+an^2\varepsilon^2}{n-1+\varepsilon^2n^2}+\frac{\beta\lambda(p-1)}{\alpha^2}+\frac{\beta\lambda p}{n\alpha^2}+\frac{a}{n-1+n^2\varepsilon^2}\right]\int v^{a-1}Q^{p-2}|\nabla v|^2 X^i_i$$
$$+\left[\frac{n(n-1)}{n-1+n^2\varepsilon^2}+\lambda(p-1-q)\right]\int v^a|\nabla v|^2 Q^{2p-4}$$
$$+\frac{n}{n-1+n^2\varepsilon^2}\int v^a(F^i_j+ML^i_j)(F^j_i+ML^j_i)$$
$$+\left[-\frac{1}{4}\left(\frac{\beta\lambda p}{\alpha^2}+\frac{na}{n-1+n^2\varepsilon^2}\right)^2\frac{(n-1+n^2\varepsilon^2)}{n}\frac{(n-1)}{n}\right.$$
$$+\frac{\beta\lambda(p-1)(a-1)}{\alpha^2}$$
$$\left.-(k+a)(\beta+1)(p-1)\right]\int v^{a-2}|\nabla v|^4 Q^{2p-4}. \tag{3.2}$$

Here we have only four terms but three parameters β, a, ε, we shall choose them properly to cancel three terms.

First, we choose ε to make

$$\frac{n(n-1)}{n-1+n^2\varepsilon^2}+\lambda(p-1-q)=0, \tag{3.3}$$

then from (3.3) we know the coefficient of the second term in identity (3.2) is zero.

To see this is possible, we show that

Lemma 3.1 *If the constants p, q, α and λ satisfy the conditions in the Theorem 1.2, then we have $n+\lambda(p-1-q)>0$.*

Proof Recall that $\lambda=\alpha(n+1-\alpha q)$ and $\alpha=\frac{p}{q+1-p}$, then we need to show

$$n-p\left(n+1-\frac{pq}{q+1-p}\right)>0,$$

which holds iff

$$\frac{pq}{q+1-p} > \frac{np+p-n}{p}.$$

The above inequality is reduced to

$$(1-p)(n-p)q > (1-p)(np+p-n),$$

which is correct since q is subcritical,

$$q < \frac{(n+1)(p-1)+1}{n-p}.$$

□

Now we take $\varepsilon = [\frac{n-\lambda(q+1-p)}{\lambda(q+1-p)}]^{\frac{1}{2}}(n-1)^{\frac{1}{2}}n^{-1}$ then we get

$$n^2\varepsilon^2 = \frac{[n-\lambda(q+1-p)]}{\lambda(q+1-p)}(n-1), \tag{3.4}$$

and

$$\frac{n}{n-1+n^2\varepsilon^2} = \frac{\lambda(q+1-p)}{n-1}. \tag{3.5}$$

Second, we set $a = t\beta$, and take t to make

$$\frac{\lambda p}{n\alpha^2} - q + \frac{tn^2\varepsilon^2}{n-1+n^2\varepsilon^2} + \frac{\lambda(p-1)}{\alpha^2} = 0. \tag{3.6}$$

From (3.6), we eliminate the first term in the identity (3.2).
By substituting ε, we take

$$t = \left(q - \frac{\lambda(p-1)}{\alpha^2} - \frac{\lambda p}{n\alpha^2}\right)\frac{n}{n-\lambda(q+1-p)}. \tag{3.7}$$

Now we simplify (3.7).

Lemma 3.2 *In fact* $t = \frac{n+1}{\alpha}$.

Proof First we have

$$\begin{aligned} n-\lambda(q+1-p) &= \frac{1}{q+1-p}\left[(q+1-p)n - p(q+1-p)(n+1) + p^2q\right] \\ &= \frac{1}{q+1-p}\Big[qn + (1-p)n - (p-1)(q+1-p)(n+1) \end{aligned}$$

$$- (q+1-p)(n+1) + q + (p^2-1)q\Big]$$
$$= \frac{p-1}{q+1-p}\Big[1-(q+1-p)(n+1)+(p+1)q\Big].$$

And we can get

$$\begin{aligned}
& n\left(q - \frac{\lambda(p-1)}{\alpha^2} - \frac{\lambda p}{n\alpha^2}\right) \\
&= nq - \frac{n(n+1-\alpha q)(p-1)}{\alpha} - \frac{(n+1-\alpha q)p}{\alpha} \\
&= nq - \frac{n+1-\alpha q}{\alpha} - \frac{n(n+1-\alpha q)(p-1)}{\alpha} - \frac{(n+1-\alpha q)(p-1)}{\alpha} \\
&= \frac{nq\alpha - n - 1 + \alpha q}{\alpha} - \frac{(n+1-\alpha q)(p-1)(n+1)}{\alpha} \\
&= \frac{(q\alpha-1)(n+1)}{\alpha} - \frac{(n+1-\alpha q)(p-1)(n+1)}{\alpha} \\
&= \frac{n+1}{\alpha}\Big[q\alpha - 1 - (n+1-\alpha q)(p-1)\Big] \\
&= \frac{n+1}{\alpha}\left[\frac{(q+1)(p-1)}{q+1-p} - (n+1-\alpha q)(p-1)\right] \\
&= \frac{n+1}{\alpha}\frac{(p-1)}{(q+1-p)}\Big[q+1-(n+1)(q+1-p)+pq\Big].
\end{aligned}$$

□

Now we substitute ε and $a = \frac{n+1}{\alpha}\beta$ into the coefficient of $\int v^{a-2}|\nabla v|^4 Q^{2p-4}$, and we shall find β such that the coefficient of the last term in the identity (3.2) also vanishes.

We set the coefficient of $\int v^{a-2}|\nabla v|^4 Q^{2p-4}$ in (3.2) equals $g(\beta)$, where

$$\begin{aligned}
g(\beta) &= -\frac{1}{4}\left(\frac{\beta\lambda p}{\alpha^2} + \frac{na}{n-1+n^2\varepsilon^2}\right)^2 \frac{(n-1+n^2\varepsilon^2)}{n}\frac{(n-1)}{n} + \frac{\beta\lambda(p-1)(a-1)}{\alpha^2} \\
&\quad - (k+a)(\beta+1)(p-1) \\
&= -\frac{1}{4n}\left(\frac{\lambda p}{\alpha^2} + \frac{n+1}{\alpha}\frac{\lambda(q+1-p)}{n-1}\right)^2 \frac{(n-1)^2}{\lambda(q+1-p)}\beta^2 + \frac{\lambda(p-1)}{\alpha^2}\frac{(n+1)}{\alpha}\beta^2 \\
&\quad - (p-1)^2\beta^2 + q(p-1)\beta^2 - \frac{(n+1)(p-1)}{\alpha}\beta^2 \\
&\quad + q(p-1)\beta - \frac{(n+1)(p-1)}{\alpha}\beta - \frac{\lambda(p-1)}{\alpha^2}\beta - 2(p-1)^2\beta \\
&\quad - (p-1)^2.
\end{aligned}$$

For this quadratic function, we have the following lemma.

Lemma 3.3 $\exists!\beta_0$, *such that* $g(\beta_0) = 0$.

Proof Since $g(\beta)$ is a quadratic function, we show that its determinant identically vanishes, which is

$$\begin{aligned}
&\left[-\frac{\lambda}{\alpha^2}-2(p-1)+q-\frac{n+1}{\alpha}\right]^2(p-1)^2\\
&+4(p-1)^2\left[-\frac{1}{4n}\left(\frac{\lambda p}{\alpha^2}+\frac{n+1}{\alpha}\frac{\lambda(q+1-p)}{n-1}\right)^2\frac{(n-1)^2}{\lambda(q+1-p)}+\frac{\lambda(p-1)}{\alpha^2}\frac{n+1}{\alpha}\right.\\
&\left.+q(p-1)-(p-1)^2-\frac{(n+1)(p-1)}{\alpha}\right]=0. \qquad (3.8)
\end{aligned}$$

We simplify it term by term, first we have

$$\begin{aligned}
&-\frac{\lambda}{\alpha^2}-2(p-1)+q-\frac{n+1}{\alpha}\\
&=\frac{-(n+1)(q+1-p)+pq}{p}-2(p-1)+q-\frac{(n+1)(q+1-p)}{p}\\
&=-\frac{2(n+1)(q+1-p)}{p}+2(q+1-p)\\
&=\frac{2(p-n-1)(q+1-p)}{p}.
\end{aligned}$$

Second,

$$\begin{aligned}
&\frac{\lambda p}{\alpha^2}+\frac{(n+1)}{\alpha}\frac{\lambda(q+1-p)}{n-1}\\
&\quad=(n+1-\alpha q)(q+1-p)\left[1+\frac{n+1}{n-1}\right]\\
&\quad=(n+1-\alpha q)(q+1-p)\frac{2n}{n-1}.
\end{aligned}$$

We compute

$$\begin{aligned}
&-\frac{1}{4n}\left(\frac{\lambda p}{\alpha^2}+\frac{n+1}{\alpha}\frac{\lambda(q+1-p)}{n-1}\right)^2\frac{(n-1)^2}{\lambda(q+1-p)}\\
&=-\frac{1}{4n}(n+1-\alpha q)^2(q+1-p)^2\frac{4n^2}{(n-1)^2}\frac{(n-1)^2}{\lambda(q+1-p)}\\
&=-\frac{n(n+1-\alpha q)(q+1-p)^2}{p}.
\end{aligned}$$

Third,

$$\frac{\lambda(p-1)(n+1)}{\alpha^3}=\frac{(n+1-\alpha q)(p-1)(n+1)(q+1-p)^2}{p^2}.$$

Last,

$$-\frac{(n+1)(p-1)}{\alpha} = -\frac{(n+1)(p-1)(q+1-p)}{p}.$$

After some computations, (3.8) is equivalent to

$$\begin{aligned} &\frac{(p-n-1)^2(q+1-p)^2}{p^2} - \frac{n(n+1-\alpha q)(q+1-p)^2}{p} \\ &+ \frac{(n+1-\alpha q)(p-1)(n+1)(q+1-p)^2}{p^2} \\ &- \frac{(n+1)(p-1)(q+1-p)}{p} + (q+1-p)(p-1) = 0. \end{aligned}$$

Multiplying $\frac{p^2}{q+1-p}$ and using $(n+1-\alpha q)(q+1-p) = (q+1-p)(n+1) - pq$, it is equivalent for us to show

$$\begin{aligned} &(p-n-1)^2(q+1-p) - pn(q+1-p)(n+1) + p^2n(q+1-p) + p^2n(p-1) \\ &\quad + (n+1)(p-1)(n+1)(q+1-p) - (n+1)(p-1)p(q+1-p) \\ &\quad + (n+1)(p-1)p(1-p) - p(p-1)(n+1) + p^2(p-1) = 0, \end{aligned}$$

which holds iff

$$\begin{aligned} &(q+1-p)[(p-1-n)^2 - pn(n+1) + p^2n + (n+1)^2(p-1) \\ &\quad + (n+1)(p-1)p] = 0, \end{aligned}$$

which is correct by direct computation.

In fact,

$$g(\beta) = -\frac{(n+1-p)^2}{\alpha^2}\beta^2 + \frac{2(p-1)[p-(n+1)]}{\alpha}\beta - (p-1)^2.$$

We can choose

$$\beta_0 = -\frac{p(p-1)}{(n+1-p)(q+1-p)}$$

□

After taking β_0 such that $g(\beta_0) = 0$, then the first term, the second term and the last term in (3.2) are canceled. At last we get the following result.

Proposition 3.4 *If v is a positive solution of the Eq.* (2.6)*, then for the above determined constants ε, β, a we have*

$$0 = \frac{n}{n-1+n^2\varepsilon^2}\int v^a(F^i_j + ML^i_j)(F^j_i + ML^j_i). \qquad (3.9)$$

To deduce the desired results, we cite a key Lemma from [1] or [6].

Lemma 3.5 *Let the matrix A be symmetric with positive eigenvalues and let λ_{min} and λ_{max} be its smallest and largest eigenvalue, respectively; let B be a symmetric matrix, then*

$$trace(AB(AB)^T) \leq n \left(\frac{\lambda_{max}}{\lambda_{min}}\right)^2 trace((AB)^2).$$

Now we show

Lemma 3.6 $F^i_j + ML^i_j = (AB)_{ij}$ *where A, B satisfy the conditions of the above Lemma.*

Proof From the definition of F^i_j, L^i_j in the beginning of Sect. 2, we have

$$\begin{aligned}
F^i_j + ML^i_j &= (Q^{p-2}v_i)_j + \left(\varepsilon - \frac{1}{n}\right) X^l_l g_{ij} + M\frac{v_i v_j}{v} Q^{p-2} - \frac{M|\nabla v|^2}{nv} Q^{p-2} g_{ij} \\
&= Q^{p-4}[(p-2)\beta^2 v_l v_j v_{il} + (\alpha^2 v^2 + \beta^2 |\nabla v|^2) v_{ij}] \\
&\quad + (p-2) Q^{p-4} \alpha^2 v v_i v_j + \left(\varepsilon - \frac{1}{n}\right) X^l_l g_{ij} + M\frac{v_i v_j}{v} Q^{p-2} \\
&\quad - \frac{M|\nabla v|^2}{nv} Q^{p-2} g_{ij} \\
&= (N_1 + N_2)_{ij},
\end{aligned}$$

where $(N_1)_{ij} = Q^{p-4}[(p-2)\beta^2 v_l v_j v_{il} + (\alpha^2 v^2 + \beta^2|\nabla v|^2) v_{ij}]$.

We rewrite

$$N_1 = N_3 N_4,$$

where $(N_4)_{ij} = Q^{p-2} v_{ij}$, $(N_3)_{ij} = (p-2)\frac{\beta^2|\nabla v|^2}{\alpha^2 v^2 + \beta^2 |\nabla v|^2} \frac{v_i v_j}{|\nabla v|^2} + \delta_{ij}$, N_3 is positive define with eigenvalues 1 and $1 + (p-2)\frac{\beta^2|\nabla v|^2}{\alpha^2 v^2 + \beta^2|\nabla v|^2}$. From basic linear algebra we have

$$(N_3^{-1})_{ij} = \delta_{ij} - (p-2)\frac{\beta^2|\nabla v|^2}{\alpha^2 v^2 + (p-1)\beta^2|\nabla v|^2} \frac{v_i v_j}{|\nabla v|^2}.$$

Then

$$N_1 + N_2 = N_3(N_4 + N_3^{-1} N_2).$$

By direct calculations, $N_3^{-1} N_2$ is also a symmetric matrix.

Setting $A = N_3$, $B = N_4 + N_3^{-1} N_2$, we have done. □

Now we prove the following last lemma.

Lemma 3.7 $|\nabla v| = 0.$

Proof By Lemmas 3.5, 3.6 and Proposition 3.4 we have

$$F^i_j + ML^i_j = 0,$$

which is

$$E^i_j + \varepsilon X^l_l g_{ij} + ML^i_j = 0.$$

By taking trace we have

$$X^l_l = 0.$$

Then

$$0 = \int X^i_i X^j_j = \frac{n}{n-1}\int E^i_j E^j_i + \frac{n}{n-1}\int R_{ij} v_i v_j Q^{2p-4}.$$

Following the method of Lemma 3.6, one can show that $\int E^i_j E^j_i \geq 0$, which forces that $|\nabla v| = 0$. Then we get v is constant and complete the proof of Theorem1.2. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

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